

ON THE COMPUTATION OF THE PICARD GROUP FOR $K3$ SURFACES[‡]

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1. INTRODUCTION

1.1. In this note, we will present a method to construct examples of $K3$ surfaces of geometric Picard rank 1. Our approach is a refinement of that of R. van Luijk [vL]. It is based on an analysis of the Galois module structure on étale cohomology. This allows to abandon the original limitation to cases of Picard rank 2 after reduction modulo p . Furthermore, the use of Galois data enables us to construct examples which require significantly less computation time.

1.2. The Picard group of a $K3$ surface S is a highly interesting invariant. In general, it is isomorphic to \mathbb{Z}^n for some $n = 1, \dots, 20$. The first explicit examples of $K3$ surfaces over \mathbb{Q} with geometric Picard rank 1 were constructed by R. van Luijk [vL]. His method is based on reduction modulo p . It works as follows.

- i) At a place p of good reduction, the Picard group $\text{Pic}(S_{\overline{\mathbb{Q}}})$ of the surface injects into the Picard group $\text{Pic}(S_{\overline{\mathbb{F}}_p})$ of its reduction modulo p .
- ii) On its part, $\text{Pic}(S_{\overline{\mathbb{F}}_p})$ injects into the second étale cohomology group $H_{\text{ét}}^2(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$.
- iii) Only roots of unity can arise as eigenvalues of the Frobenius on the image of $\text{Pic}(S_{\overline{\mathbb{F}}_p})$ in $H_{\text{ét}}^2(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$. The number of eigenvalues of this form is therefore an upper bound for the Picard rank of $S_{\overline{\mathbb{F}}_p}$. We can compute the eigenvalues of Frob by counting the points on S , defined over \mathbb{F}_p and some finite extensions.

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Doing this for one prime, one obtains an upper bound for $\text{rk Pic}(S_{\mathbb{F}_p})$ which is always even. The Tate conjecture asserts that this bound is actually sharp.

When one wants to prove that the Picard rank over $\overline{\mathbb{Q}}$ is, in fact, equal to 1, the best which could happen is to find a prime that yields an upper bound of 2. There is not much hope to do better when working with a single prime.

iv) In this case, the assumption that the surface would have Picard rank 2 over $\overline{\mathbb{Q}}$ implies that the discriminants of both Picard groups, $\text{Pic}(S_{\overline{\mathbb{Q}}})$ and $\text{Pic}(S_{\mathbb{F}_p})$, are in the same square class. Note here that reduction modulo p respects the intersection product.

v) When one combines information from two primes, it may happen that we get the rank bound 2 at both places but different square classes for the discriminant do arise. Then, these data are incompatible with Picard rank 2 over $\overline{\mathbb{Q}}$.

On the other hand, there is a non-trivial divisor known explicitly. Altogether, rank 1 is proven.

Remark 1.3. This method has been applied by several authors in order to construct $K3$ surfaces with prescribed Picard rank [vL, Kl, EJ1].

1.4. The refinement. In this note, we will refine van Luijk's method. Our idea is the following. We do not look at the ranks, only. We analyze the Galois module structures on the Picard groups, too. The point here is that a Galois module typically has submodules by far not of every rank.

As an example, we will construct $K3$ surfaces of geometric Picard rank 1 such that the reduction modulo 3 has geometric Picard rank 4 and the reduction modulo 5 has geometric Picard rank 14.

Remark 1.5. This work continues our investigations on Galois module structures on the Picard group. In [EJ2, EJ3, EJ4], we constructed cubic surfaces S over \mathbb{Q} with prescribed Galois module structure on $\text{Pic}(S)$.

2. THE PICARD GROUP AS A GALOIS MODULE

2.1. Let K be a field and S an algebraic surface defined over K . Denote by S the \mathbb{Q} -vector space $\text{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. On S , there is a natural $\text{Gal}(\overline{K}/K)$ -operation. The kernel of this representation is a normal subgroup of finite index. It corresponds to a finite Galois extension L of K . In fact, we have a $\text{Gal}(L/K)$ -representation.

The group $\text{Gal}(\overline{K}/L)$ acts trivially on $\text{Pic}(S_{\overline{K}})$. I.e.,

$$\text{Pic}(S_{\overline{K}}) = \text{Pic}(S_{\overline{K}})^{\text{Gal}(\overline{K}/L)}.$$

Within this, $\text{Pic}(S_L)$ is, in general, a subgroup of finite index. Equality is true under the hypothesis that $S(L) \neq \emptyset$.

2.2. Now suppose K is a number field and \mathfrak{p} is a prime ideal of K . We will denote the residue class field by k . Further, let S be a $K3$ surface over K with good reduction at \mathfrak{p} . There is an injection of $\text{Pic}(S_{\overline{K}})$ into $\text{Pic}(S_{\overline{k}})$. Taking the tensor product, this yields an inclusion of vector spaces $\text{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \text{Pic}(S_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Both spaces are equipped with a Galois operation. On $\text{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, we have a $\text{Gal}(L/K)$ -action. On $\text{Pic}(S_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, only $\text{Gal}(\overline{k}/k) = \langle \text{Frob} \rangle$ operates.

Lemma 2.3. *The field extension L/K is unramified at \mathfrak{p} .*

Proof. Let $D \in \text{Div}(S_L)$ be an arbitrary divisor. By good reduction, D extends to a divisor on a smooth model \mathcal{S} over the integer ring \mathcal{O}_L . In particular, we have the reduction $D_{\mathfrak{q}}$ of D on the special fiber $S_{\mathfrak{q}}$. Here, \mathfrak{q} is any prime, lying above \mathfrak{p} .

$\mathcal{O}_L/\mathfrak{q}$ is a finite extension of k . Correspondingly, there is an unramified extension $L' \subset L$ of K . Good reduction implies that every divisor on $S_{\mathfrak{q}}$ lifts to $S_{L'}$. Consequently, we have a divisor $D' \in \text{Div}(S_{L'})$ which has the same reduction as D .

As intersection products are respected by reduction, we see that the intersection number of $D'_L - D$ with any divisor is zero. The standard argument from [BPV, Proposition VIII.3.6.i)] implies $D'_L - D = 0$. In other words, D is defined over an unramified extension. \square

2.4. There is a Frobenius lift to L which is unique up to conjugation. When we choose a particular prime \mathfrak{q} , lying above \mathfrak{p} , we fix a concrete Frobenius lift. Then, $\text{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ becomes a $\text{Gal}(\overline{k}/k)$ -submodule of $\text{Pic}(S_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.5. Computability of the Galois representation. The simplest way to understand the $\text{Gal}(\overline{k}/k)$ -representation on $\text{Pic}(S_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is to use étale cohomology. Counting the numbers of points, S has over finite extensions of k , we can compute the characteristic polynomial Φ of the Frobenius on $H_{\text{ét}}^2(S_{\overline{k}} \otimes_{\mathbb{Q}_l}(1))$. This is actually a polynomial with rational, even integer, coefficients and independent of the choice of $l \neq p$ [De, Théorème 1.6].

Denote by V_{Tate} the largest subspace of $H_{\text{ét}}^2(S_{\overline{k}} \otimes_{\mathbb{Q}_l}(1))$ on which all eigenvalues of the Frobenius are roots of unity. On the other hand, let P_{conj} be the subgroup of $\text{Pic}(S_{\overline{K}})$ generated by the conjugates of all the divisors we know explicitly.

Then, we have the following chain of $\text{Gal}(\overline{k}/k)$ -modules,

$$H_{\text{ét}}^2(S_{\overline{k}}, \mathbb{Q}_l(1)) \supseteq V_{\text{Tate}} \supseteq \text{Pic}(S_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \supseteq \text{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \supseteq P_{\text{conj}} \otimes_{\mathbb{Z}} \mathbb{Q}_l.$$

In an optimal situation, the quotient space $V_{\text{Tate}}/(P_{\text{conj}} \otimes_{\mathbb{Z}} \mathbb{Q}_l)$ has only finitely many $\text{Gal}(\overline{k}/k)$ -submodules. This finiteness condition generalizes the codimension one condition, applied in van Luijk's method, step v).

Our main strategy will then be as follows. We inspect the $\text{Gal}(\overline{k}/k)$ -submodules of $V_{\text{Tate}}/(P_{\text{conj}} \otimes_{\mathbb{Z}} \mathbb{Q}_l)$. For all these, except for

the null space, we aim to exclude the possibility that it coincides with $(\text{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l) / (P_{\text{conj}} \otimes_{\mathbb{Z}} \mathbb{Q}_l)$.

Remarks 2.6. a) A sufficient criterion for a $\text{Gal}(\overline{k}/k)$ -module to have only finitely many submodules is that the characteristic polynomial of Frob has only simple roots. This fact, although very standard, is central to our method.

b) Only submodules of the form $P \otimes_{\mathbb{Z}} \mathbb{Q}_l$ for P a $\text{Gal}(\overline{k}/k)$ -submodule of $\text{Pic}(S_{\overline{K}})$ are possible candidates for $\text{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$. Such submodules automatically lead to factors of Φ with coefficients in \mathbb{Q} .

Definition 2.7. We will call a $\text{Gal}(\overline{k}/k)$ -submodule of $H_{\text{ét}}^2(S_{\overline{K}}, \mathbb{Q}_l(1))$ *admissible* if it is a \mathbb{Q}_l -subvector space and the characteristic polynomial of Frob has rational coefficients.

Remark 2.8. In some sense, we apply the van der Waerden criterion to the representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Picard group and étale cohomology.

Remark 2.9. In the practical computations presented below, we will work with \mathbb{Q}_l instead of $\mathbb{Q}_l(1)$. This is the canonical choice from the point of view of counting points but not for the image of the Picard group. The relevant zeroes of the characteristic polynomial of the Frobenius are then those of the form q times a root of unity.

3. AN EXAMPLE

3.1. Formulation.

Example 3.1.1. Let $S: w^2 = f_6(x, y, z)$ be a K3 surface of degree 2 over \mathbb{Q} . Assume the congruences

$$f_6 \equiv y^6 + x^4 y^2 - 2x^2 y^4 + 2x^5 z + 3xz^5 + z^6 \pmod{5}$$

and

$$\begin{aligned} f_6 \equiv & 2x^6 + x^4 y^2 + 2x^3 y^2 z + x^2 y^2 z^2 + x^2 y z^3 + 2x^2 z^4 \\ & + xy^4 z + xy^3 z^2 + xy^2 z^3 + 2xz^5 + 2y^6 + y^4 z^2 + y^3 z^3 \pmod{3}. \end{aligned}$$

Then, S has geometric Picard rank 1.

3.2. Explicit divisors.

Notation 3.2.1. We will write $\text{pr}: S \rightarrow \mathbf{P}^2$ for the canonical projection. On S , there is the ample divisor $H := \pi^* L$ for L a line on \mathbf{P}^2 .

3.2.2. Let C be any irreducible divisor on S . Then, $D := \pi_* C$ is a curve in \mathbf{P}^2 . We denote its degree by d . The projection from C to D is generically 2:1 or 1:1. In the case it is 2:1, we have $C = \pi^* D \sim dH$.

Thus, to generate a Picard group of rank >1 , divisors are needed which are generically 1:1 over their projections. This means, $\pi^* D$ must be reducible into two components which we call the *splittings* of D .

A divisor D has a split pull-back if and only if f_6 is a perfect square on (the normalization of) D . A necessary condition is that the intersection of D with the ramification locus is a 0-cycle divisible by 2.

3.3. The Artin-Tate conjecture.

3.3.1. The Picard group of a projective variety is equipped with a \mathbb{Z} -valued bilinear form, the intersection form. Therefore, associated to $\text{Pic}(S_{\bar{k}})$, we have its discriminant, an integer. The same applies to every subgroup of $\text{Pic}(S_{\bar{k}})$.

For a \mathbb{Q}_l -vector space contained in $\text{Pic}(S_{\bar{k}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$, the discriminant is determined only up to a factor being a square in \mathbb{Q}_l . However, every non-square in \mathbb{Q} is a non-square in \mathbb{Q}_l for some suitable prime $l \gg 0$.

3.3.2. Let us recall the Artin-Tate conjecture in the special case of a $K3$ surface.

Conjecture (Artin-Tate) . *Let Y be a $K3$ surface over \mathbb{F}_q . Denote by ρ the rank and by Δ the discriminant of the Picard group of Y , defined over \mathbb{F}_q . Then,*

$$(1) \quad \lim_{T \rightarrow q} \frac{\Phi(T)}{(T - q)^\rho} = q^{21-\rho} \# \text{Br}(Y) |\Delta|.$$

Here, Φ is the characteristic polynomial of the Frobenius on $H_{\text{ét}}^2(Y_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)$. Finally, $\text{Br}(Y)$ denotes the Brauer group of Y .

Remarks 3.3.3. a) The Artin-Tate conjecture allows to compute the square class of the discriminant of the Picard group over a finite field without any knowledge of explicit generators.

b) Observe that $\# \text{Br}(Y)$ is always a perfect square [LLR].

c) The Artin-Tate conjecture is proven for most $K3$ surfaces. Most notably, the Tate conjecture implies the Artin-Tate conjecture [Mi2]. We will use the Artin-Tate conjecture only in situations where the Tate conjecture is true. Thus, our final result will not depend on unproven statements.

3.4. The modulo 3 information.

3.4.1. The sextic curve given by “ $f_6 = 0$ ” has three conjugate conics, each tangent in six points. Indeed, note that, for

$$f_3 := x^3 + 2x^2y + x^2z + 2xy^2 + xyz + xz^2 + y^3 + y^2z + 2yz^2 + 2z^3,$$

the term $f_6 - f_3^2$ factors into three quadratic forms over \mathbb{F}_{27} . Consequently, we have three divisors on $\mathbf{P}_{\mathbb{F}_{27}}^2$ the pull-backs of which split.

3.4.2. Counting the points on S over \mathbb{F}_{3^n} for $n = 1, \dots, 11$ yields the numbers $-2, -8, 28, 100, 388, 2\,458, 964, -692, 26\,650, -20\,528$, and $-464\,444$ as the traces of the iterated Frobenius on $H_{\text{ét}}^2(S_{\bar{\mathbb{F}}_3}, \mathbb{Q}_l)$. Taking into account the

fact that p is a root of the characteristic polynomial Φ , these data determine this polynomial uniquely,

$$\begin{aligned}\Phi(t) = & t^{22} + 2t^{21} + 6t^{20} - 27t^{18} - 162t^{17} - 729t^{16} - 1458t^{15} - 2187t^{14} \\ & + 19683t^{12} + 118098t^{11} + 177147t^{10} - 1594323t^8 - 9565938t^7 \\ & - 43046721t^6 - 86093442t^5 - 129140163t^4 + 2324522934t^2 \\ & + 6973568802t + 31381059609.\end{aligned}$$

The functional equation holds with the plus sign. We factorize and get

$$\begin{aligned}\Phi(t) = & (t-3)^2(t^2+3t+9) \\ & (t^{18} + 5t^{17} + 21t^{16} + 90t^{15} + 297t^{14} + 891t^{13} + 2673t^{12} + 7290t^{11} \\ & + 19683t^{10} + 59049t^9 + 177147t^8 + 590490t^7 + 1948617t^6 \\ & + 5845851t^5 + 17537553t^4 + 47829690t^3 + 100442349t^2 \\ & + 215233605t + 387420489).\end{aligned}$$

3.4.3. From this, we derive an upper bound of 4 for the rank of the Picard group. In the notation of section 2, V_{Tate} is a \mathbb{Q}_l -vector space of dimension four. On the other hand, P_{conj} is generated by H . As H corresponds to one of the factors $(t-3)$, the characteristic polynomial of the Frobenius on $V_{\text{Tate}}/(P_{\text{conj}} \otimes_{\mathbb{Z}} \mathbb{Q}_l)$ is $(t-3)(t^2+3t+9)$. It has only simple roots.

Consequently, for each of the dimensions 1, 2, 3, and 4, there is precisely one admissible $\text{Gal}(\overline{\mathbb{F}_3}/\mathbb{F}_3)$ -submodule in $H_{\text{ét}}^2(S_{\overline{\mathbb{F}_3}}, \mathbb{Q}_l)$ containing the Chern class of H .

3.4.4. Let us compute the corresponding discriminants.

i) In the one-dimensional case, we have discriminant 2.

ii) It turns out that splitting the divisors given by the conics which are six times tangent yields a rank three submodule M of $\text{Pic}(S_{\overline{\mathbb{F}_3}})$. Its discriminant is

$$\text{disc } M = \det \begin{pmatrix} -2 & 6 & 0 \\ 6 & -2 & 4 \\ 0 & 4 & -2 \end{pmatrix} = 96.$$

Hence, in the three-dimensional case, the discriminant is in the square class of 6.

iii) For the case of dimension four, we may suppose that $\text{Pic}(S_{\overline{\mathbb{F}_3}})$ is of rank four. As $\text{Gal}(\overline{\mathbb{F}_3}/\mathbb{F}_{27})$ acts trivially on $\text{Pic}(S_{\overline{\mathbb{F}_3}})$, the group $\text{Pic}(S_{\mathbb{F}_{27}})$ is of rank four, already. This means, the Tate conjecture is true for $S_{\mathbb{F}_{27}}$.

We may compute the square class of the corresponding discriminant according to the Artin-Tate conjecture. The result is (-163) .

iv) Finally, consider the two-dimensional case. We may suppose that $\text{Pic}(S_{\overline{\mathbb{F}_3}})$ has a $\text{Gal}(\overline{\mathbb{F}_3}/\mathbb{F}_3)$ -submodule N which is of rank two and contains H .

The corresponding characteristic polynomial of the Frobenius is, necessarily, equal to $(t - 3)^2$.

On the other hand, there is the rank three submodule M generated by the splittings of the conics which are six times tangent. As the corresponding factors are $(t - 3)(t^2 + 3t + 9)$, the modules N and M together generate rank 4. The Tate conjecture is true for $S_{\mathbb{F}_{27}}$. Consequently, it is true for $S_{\mathbb{F}_3}$, too.

Using the Artin-Tate conjecture, we can compute the square class of the discriminant. It turns out to be (-489) .

Remark 3.4.5. The Tate conjecture predicts Picard rank 2 for $S_{\mathbb{F}_3}$. Let C be an irreducible divisor linearly independent of H . Then, C is a splitting of a curve D of degree $d \geq 23$. Indeed, H is a genus 2 curve. Hence, $H^2 = 2$. For the discriminant, we find $-489 \geq 2C^2 - d^2$. As $C^2 \geq -2$, the assertion follows.

Further, D is highly singular on the ramification locus. In fact, we have $C^2 \leq \frac{d^2 - 489}{2}$ and $D^2 = d^2$. Hence, going from D to C lowers the arithmetic genus by at least $\frac{d^2 + 489}{4}$.

3.5. The modulo 5 information.

3.5.1. The sextic curve given by “ $f_6 = 0$ ” has six tritangent lines. These are given by $L_a : t \mapsto [1 : t : a]$ where a is a zero of $a^6 + 3a^5 + 2a$. The pull-back of each of these lines splits on the $K3$ surface $S_{\mathbb{F}_5}$.

3.5.2. On the other hand, counting points yields the following traces of the iterated Frobenius on $H_{\text{ét}}^2(S_{\mathbb{F}_5}, \mathbb{Q}_l)$,

$$15, 95, -75, 2\,075, -1\,250, -14\,875, 523\,125, 741\,875, 853\,125, 11\,293\,750.$$

This leads to the characteristic polynomial

$$\begin{aligned} \Phi(t) &= t^{22} - 15t^{21} + 65t^{20} + 175t^{19} - 3000t^{18} + 11437t^{17} + 10630t^{16} \\ &\quad - 385950t^{15} + 2445250t^{14} - 4530625t^{13} - 38478125t^{12} \\ &\quad + 305656250t^{11} - 566328125t^{10} - 4809765625t^9 + 38207031250t^8 \\ &\quad - 101308593750t^7 - 143457031250t^6 + 2792236328125t^5 \\ &\quad - 14189453125000t^4 + 16400146484375t^3 + 247955322265625t^2 \\ &\quad - 1430511474609375t + 2384185791015625 \\ &= (t - 5)^2(t^4 + 5t^3 + 25t^2 + 125t + 625) \\ &\quad (t^8 - 5t^7 + 125t^5 - 625t^4 + 3\,125t^3 - 78\,125t + 390\,625) \\ &\quad (t^8 - 5t^7 - 10t^6 + 75t^5 - 125t^4 + 1\,875t^3 - 6\,250t^2 - \dots \\ &\quad \dots - 78\,125t + 390\,625). \end{aligned}$$

Observe here the first two factors correspond to the part of the Picard group generated by the splittings of the six tritangent lines. They could have been computed directly from the intersection matrix of these divisors.

Remark 3.5.3. The knowledge of these two factors allows to compute the characteristic polynomial only from the numbers of points over $\mathbb{F}_5, \dots, \mathbb{F}_{5^8}$. Counting them takes approximately five minutes when one uses the method described in [EJ1, Algorithm 15].

3.5.4. Here, V_{Tate} is a vector space of dimension 14. Again, P_{conj} is generated by H . The characteristic polynomial of the Frobenius on $V_{\text{Tate}}/(P_{\text{conj}} \otimes_{\mathbb{Z}} \mathbb{Q}_l)$ is

$$(t - 5)(t^4 + 5t^3 + 25t^2 + 125t + 625) \\ (t^8 - 5t^7 + 125t^5 - 625t^4 + 3 \cdot 125t^3 - 78 \cdot 125t + 390 \cdot 625)$$

having only simple roots. This shows that, in each of the dimensions 1, 2, 5, 6, 9, 10, 13, and 14, there is precisely one admissible $\text{Gal}(\overline{\mathbb{F}_5}/\mathbb{F}_5)$ -submodule in $H_{\text{ét}}^2(S_{\overline{\mathbb{F}_5}}, \mathbb{Q}_l)$ containing the Chern class of H .

3.5.5. For the cases of low rank, let us compute the square classes of the discriminant.

- i) In the one-dimensional case, we have discriminant 2.
- ii) For the two-dimensional case, recall that we know six tritangent lines of the ramification locus. One of them, L_0 , is defined over \mathbb{F}_5 . Splitting π^*L_0 yields rank two alone. For the discriminant, we find

$$\det \begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix} = -5.$$

Remark 3.5.6. Using the Artin-Tate conjecture, we may compute conditional values for the square classes of the discriminant for the 6- and 14-dimensional modules. Both are actually equal to (-1) .

3.6. The situation over $\overline{\mathbb{Q}}$.

3.6.1. Proof of 3.1.1. Now we can put everything together and show that the K3 surfaces described in Example 3.1.1 indeed have geometric Picard rank 1.

The Picard group $\text{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ injects as a Galois submodule into the second étale cohomology groups $H_{\text{ét}}^2(S_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)$ for $p = 3$ and 5. The modulo 3 data show that this module has \mathbb{Q}_l -dimension 1, 2, 3 or 4. The reduction modulo 5 allows the \mathbb{Q}_l -dimensions 1, 2, 5, 6, 9, 10, 13 and 14. Consequently, the Picard rank is either 1 or 2.

To exclude the possibility of rank 2, we compare the discriminants. The reduction modulo 3 enforces discriminant (-489) while the reduction modulo 5 yields discriminant (-5) . This is a contradiction, e.g., for $l = 17$. \square

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